1. Let $ABCDEFGH$ be a unit cube such that $ABCD$ is one face of the cube and $\overline{AE}$, $\overline{BF}$, $\overline{CG}$, and $\overline{DH}$ are all edges of the cube. Points I, J, K, and L are the respective midpoints of $\overline{AF}$, $\overline{BG}$, $\overline{CH}$, and $\overline{DE}$. The inscribed circle of $IJKL$ is the largest cross-section of some sphere. Compute the volume of this sphere.

   Answer: $\frac{\sqrt{2}}{24}\pi$

   Solution: Note that I, J, K, L are the centers of the faces $ABFE$, $BCGF$, $CDHG$, and $DAEH$, respectively. Thus, $IJKL$ is a square, and its side length is the hypotenuse of an isosceles right triangle with side length $\frac{1}{2}$, so it is $\frac{\sqrt{2}}{2}$. This means that the inscribed circle of $IJKL$ will have a radius which is one half the side length of $IJKL$, so the radius is $\frac{\sqrt{2}}{4}$. Thus, the sphere also has radius $\frac{\sqrt{2}}{4}$, and hence its volume is $\boxed{\frac{\sqrt{2}}{24}\pi}$.

2. Let $ABCD$ be a unit square. Points E and F are chosen on line segments $\overline{BC}$ and $\overline{CD}$, respectively, such that the area of $ABEFD$ is three times the area of triangle $\triangle ECF$. Compute the maximum possible area of triangle $\triangle AEF$.

   Answer: $\frac{1}{2}$

   Solution: Let $BE = x$ and $DF = y$. Since the areas of $ABEFD$ and ECF sum to the area of the entire square $ABCD$, which is 1, we have that $[ECF] = \frac{1}{4}$. That is, $\frac{1}{2}(1-x)(1-y) = \frac{1}{4}$. Rearranging this equation gives $x + y = \frac{1}{2} + xy$. Now, note that

$$\begin{aligned}[AEF] &= [ABEFD] - [ABE] - [ADF] \\ &= \frac{3}{4} - \frac{1}{2}\cdot 1\cdot x - \frac{1}{2}\cdot 1\cdot y \\ &= \frac{3}{4} - \frac{x+y}{2} \\ &= \frac{1}{2} - \frac{xy}{2}.\end{aligned}$$

   We now want to maximize this expression. Equivalently, we will minimize xy. Since $x, y \geq 0$, we will always have that $xy \geq 0$, so the minimum possible value is 0. This can indeed be achieved while satisfying the condition $x + y = \frac{1}{2} + xy$, for example with $x = 0$ and $y = \frac{1}{2}$. Hence the maximum possible area of $[AEF]$ is $\boxed{\frac{1}{2}}$.

3. In triangle $\triangle ABC$, M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{AC}$. Let P be the midpoint of $\overline{BN}$ and let Q be the midpoint of $\overline{CM}$. If $AM = 6$, $BC = 8$ and $BN = 7$, compute the perimeter of triangle $\triangle NPQ$.

   Answer: $\frac{17}{2}$

   Solution: Let X be the midpoint of $\overline{BM}$ and let Y be the midpoint of $\overline{CN}$.

   First, note that since M, N are the midpoints of $\overline{AB}, \overline{AC}$, respectively, we have that $\triangle AMN \sim \triangle ABC$ with the ratio of similarity being $\frac{1}{2}$. That is, $MN = \frac{1}{2}BC = 4$. Similarly, since X, P are the midpoints of $\overline{BM}, \overline{BN}$, respectively, we have that $\triangle BXP \sim \triangle BMN$ with the ratio of similarity being $\frac{1}{2}$. Thus $XP = \frac{1}{2}MN = 2$. By the same logic, we get that $QY = \frac{1}{2}MN = 2$. Also, since $\frac{AX}{AB} = \frac{AY}{AB} = \frac{3}{4}$, we have that $XY = \frac{3}{4}BC = 6$ by the similarity $\triangle AXY \sim \triangle ABC$. Hence

$$PQ = XY - XP - QY = 6 - 2 - 2 = 2.$$

Also, since $\frac{CQ}{CM} = \frac{CN}{CA} = \frac{1}{2}$, we have that $\triangle CQN \sim \triangle CMA$ with ratio of similarity $\frac{1}{2}$. Hence $NQ = \frac{1}{2}AM = 3$. Finally, since P is the midpoint of $\overline{BN}$, $NP = \frac{1}{2}BN = \frac{7}{2}$. Thus the perimeter of triangle $\triangle NPQ$ is

$$PQ + NQ + NP = 2 + 3 + \frac{7}{2} = \boxed{\frac{17}{2}}.$$