1. Define an operation $\diamondsuit$ as $a\diamondsuit b = 12a - 10b$. Compute the value of $((((20\diamondsuit 22)\diamondsuit 22)\diamondsuit 22)\diamondsuit 22)$.

   Answer: 20

   Solution: We compute $20\diamondsuit 22 = 12(20) - 10(22) = 20$. Thus, we can replace every instance of $20\diamondsuit 22$ with 20:

$$\begin{aligned}((((20\diamondsuit 22)\diamondsuit 22)\diamondsuit 22)\diamondsuit 22) &= (((20\diamondsuit 22)\diamondsuit 22)\diamondsuit 22)\\ &= ((20\diamondsuit 22)\diamondsuit 22)\\ &= 20\diamondsuit 22\\ &= \boxed{20}.\end{aligned}$$

2. The equation

$$4^x - 5 \cdot 2^{x+1} + 16 = 0$$

   has two integer solutions for x. Find their sum.

   Answer: 4

   Solution: Define $y = 2^x$. Then our equation can be rewritten as

$$y^2 - 10y + 16 = (y-2)(y-8) = 0.$$

   This gives $y = 2$ or $y = 8$, which means $x = 1$ or $x = 3$. Therefore, the sum of the solutions for x is $\boxed{4}$.

3. Suppose we have four real numbers a, b, c, d such that a is nonzero, a, b, c form a geometric sequence, in that order, and b, c, d form an arithmetic sequence, in that order. Compute the smallest possible value of $\frac{d}{a}$. (A geometric sequence is one where every succeeding term can be written as the previous term multiplied by a constant, and an arithmetic sequence is one where every succeeeding term can be written as the previous term added to a constant.)

   Answer: $-\frac{1}{8}$

   Solution: Let r be the ratio in the geometric sequence, so that $b = ar$ and $c = ar^2$. Since $d - c = c - b$, we have $d = 2c - b = a \cdot (2r^2 - r)$. The minimum of $\frac{d}{a} = 2r^2 - r$ occurs at $r = -\frac{-1}{2\cdot 2} = \frac{1}{4}$, with value $2 \cdot \left(\frac{1}{4}\right)^2 - \frac{1}{4} = \boxed{-\frac{1}{8}}$.

4. Find all real x such that

$$\lfloor x \lceil x \rceil \rfloor = 2022.$$

   Express your answer in interval notation. (Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor -4.25 \rfloor = -5$. In addition, $\lceil m \rceil$ is defined as the least integer greater than or equal to m. For example, $\lceil 2 \rceil = 2$ and $\lceil -3.25 \rceil = -3$.)

   Answer: $\left[\frac{674}{15}, \frac{2023}{45}\right)$

   Solution: Suppose x is positive. Recognize that $44^2 < 2022 < 45^2$, which implies we have $44 < x < 45$. Then, we see $\lceil x \rceil = 45$ which gives $\frac{2022}{45} \le x = \frac{674}{15} \le x$. To have an upperbound for $\lfloor 45x \rfloor = 2022$, we need $x < \frac{2023}{45}$ since any larger x will have the floor yielding 2023, violating our condition.

   Suppose x is negative. Using the same bounding from above, we see that if a solution were to exist, it must abide by $-45 < x < -44$. The $x < -44$ means $\lceil x \rceil = -44$ which forces $x < -\frac{2022}{44}$ which violates the condition that $-45 < x$.

Hence, the only solution is $x \in \boxed{\left[\frac{674}{15}, \frac{2023}{45}\right)}$.

5. For real numbers B, M, and T, we have $B^2 + M^2 + T^2 = 2022$ and $B + M + T = 72$. Compute the sum of the minimum and maximum possible values of T.

Answer: 48

Solution: From the second equation, we have that $B + M = 72 - T$. Note that $BM \leq \frac{(B+M)^2}{4}$ by AM-GM (or by rearranging $(B - M)^2 \geq 0$), so:

$$B^2 + M^2 = B^2 + M^2 + 2BM - 2BM \geq (B + M)^2 - \frac{(B + M)^2}{2} = \frac{(B + M)^2}{2} = \frac{(72 - T)^2}{2}$$

Utilizing the first equation, we now have $\frac{(72-T)^2}{2} + T^2 \leq 2022$. Rearranging this inequality, we get $3T^2 - 144T + 5184 \leq 4044$. Dividing both sides by 3 and then factoring yields $(T-10)(T-38) \leq 0$, which implies that $10 \leq T \leq 38$. Thus, the minimum possible value of T is 10 and the maximum possible value of T is 38. Their sum is $\boxed{48}$. Indeed, we find that the triples $(B, M, T) = (17, 17, 38)$ and $(B, M, T) = (31, 31, 10)$ are solutions. As a remark, notice that T is minimized and maximized when $B = M$.

6. The degree-6 polynomial f satisfies $f(7) - f(1) = 1$, $f(8) - f(2) = 16$, $f(9) - f(3) = 81$, $f(10) - f(4) = 256$ and $f(11) - f(5) = 625$. Compute $f(15) - f(-3)$.

Answer: 6723

Solution: Note that $g(x) = f(x + 6) - f(x) - x^4$ is a degree-5 polynomial whose roots are $x = 1, 2, 3, 4, 5$, so $g(x) = C(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ for some constant C and $f(x + 6) - f(x) = g(x) + x^4$. Then we have:

$$\begin{aligned} f(15) - f(-3) &= (f(15) - f(9)) + (f(9) - f(3)) + (f(3) - f(-3)) \\ &= (g(9) + 9^4) + 81 + (g(-3) + (-3)^4) \\ &= (9^4 + 2 \cdot 3^4) + (C(9 - 1)(9 - 2)(9 - 3)(9 - 4)(9 - 5) \\ &+ C(-3 - 1)(-3 - 2)(-3 - 3)(-3 - 4)(-3 - 5)) \\ &= 6723 + C \cdot (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 + (-4) \cdot (-5) \cdot (-6) \cdot (-7) \cdot (-8)) \\ &= 6723 + 0 = \boxed{6723}. \end{aligned}$$

7. Let r, s, and t be the distinct roots of $x^3 - 2022x^2 + 2022x + 2022$. Compute

$$\frac{1}{1 - r^2} + \frac{1}{1 - s^2} + \frac{1}{1 - t^2}.$$

Answer: $\mathbf{\frac{2025}{2023}}$

Solution: We can express

$$\frac{1}{1 - r^2} + \frac{1}{1 - s^2} + \frac{1}{1 - t^2} = \frac{1}{2}\left(\frac{1}{1 + r} + \frac{1}{1 + s} + \frac{1}{1 + t}\right) + \frac{1}{2}\left(\frac{1}{1 - r} + \frac{1}{1 - s} + \frac{1}{1 - t}\right).$$

We compute each subsum. First, note that the polynomial with roots $1 + r, 1 + s, 1 + t$ is

$$(x - 1)^3 - 2022(x - 1)^2 + 2022(x - 1) + 2022 = x^3 - 2025x^2 + 6069x - 2023,$$

so

$$\frac{1}{1+r}+\frac{1}{1+s}+\frac{1}{1+t}=-\frac{6069}{-2023}=3.$$

Similarly, the polynomial with roots $1-r, 1-s$, $1-t$ is

$$(1-x)^3-2022(1-x)^2+2022(1-x)+2022=-x^3-2019x^2+2019x+2023,$$

so

$$\frac{1}{1-r}+\frac{1}{1-s}+\frac{1}{1-t}=-\frac{2019}{2023},$$

resulting in our final answer of $\frac{1}{2}\left(3-\frac{2019}{2023}\right)=\boxed{\frac{2025}{2023}}$.

8. Given

$$\begin{aligned} x_1x_2\cdots x_{2022}&=1\\ (x_1+1)(x_2+1)\cdots(x_{2022}+1)&=2\\ &\vdots\\ (x_1+2021)(x_2+2021)\cdots(x_{2022}+2021)&=2^{2021}, \end{aligned}$$

compute

$$(x_1+2022)(x_2+2022)\cdots(x_{2022}+2022).$$

Answer: $2022!+2^{2022}-1$

Solution: Define

$$P(t)=(x_1+t)(x_2+t)\cdots(x_{2022}+t).$$

We are effectively given 2022 points from the problem statement and so interpolation only guarantees us a 2021 degree polynomial. From interpolation, we get

$$P(t)=\sum_{i=0}^{2021}\binom{t}{i}.$$

Suppose $Q(t)$ is the polynomial representing the value we wish to compute. We see that $Q(t)$ is monic and $Q(t)-P(t)$ must have 2022 roots (since they must agree on 2022 points) but $P(t)$ only has 2021 roots at most when interpolating. To account for this, we can rewrite $P(t)$ as

$$P(t)=(t-0)(t-1)\cdots(t-2021)+\sum_{i=0}^{2021}\binom{t}{i}.$$

The new offset term ensures we have 2022 roots and utilizes the property of Q being monic. From here, we see $P(2022)=\boxed{2022!+2^{2022}-1}$.

9. We define a sequence $x_1=\sqrt{3}$, $x_2=-1$, $x_3=2-\sqrt{3}$, and for all $n\geq 4$

$$(x_n+x_{n-3})(1-x_{n-1}^2x_{n-2}^2)=2x_{n-1}(1+x_{n-2}^2).$$

Suppose m is the smallest positive integer for which x_m is undefined. Compute m.

Answer: 10

Solution: We rewrite the recurrence to isolate

$$x_n + x_{n-3} = \frac{2x_{n-1}(1+x_{n-2}^2)}{1-x_{n-1}^2x_{n-2}^2}$$

The recurrence relation reminds us of the tangent angle addition formula, which is given by $\tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$, as it has the product of two tangents in the denominator. The squares in the denominator also motivate us to look at $\tan(a-b) = \frac{\tan a-\tan b}{1+\tan a\tan b}$. Note that if we add these two expressions together, we get

$$\begin{aligned}
&\tan(a+b)+\tan(a-b)\\
&= \frac{\tan a+\tan b}{1-\tan a\tan b}+\frac{\tan a-\tan b}{1+\tan a\tan b}\\
&= \frac{(\tan a+\tan b)(1+\tan a\tan b)+(\tan a-\tan b)(1-\tan a\tan b)}{1-\tan^2 a\tan^2 b}\\
&= \frac{\tan a+\tan b+\tan^2 a\tan b+\tan a\tan^2 b+\tan a-\tan b-\tan^2 a\tan b+\tan a\tan^2 b}{1-\tan^2 a\tan^2 b}\\
&= \frac{2\tan a+2\tan a\tan^2 b}{1-\tan^2 a\tan^2 b}
\end{aligned}$$

Note that if we set $x_n = \tan(a+b)$, $x_{n-1} = \tan a$, $x_{n-2} = \tan b$, and $x_{n-3} = \tan(a-b)$, we obtain our recurrence relation. Now all we must do is find the values of a and b. Thus, $x_1 = \tan\left(4\cdot\frac{\pi}{12}\right)$, $x_2 = \tan\left(9\cdot\frac{\pi}{12}\right)$, and $x_3 = \tan\left(13\cdot\frac{\pi}{12}\right)$. Since $\tan x$ is undefined at $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$, we would like to find the first term in the sequence $4, 9, 13, \ldots$ (where the next term is the sum of the two previous terms) that is equivalent to $6 \pmod{24}$ or $18 \pmod{24}$. Listing out terms gives us $4, 9, 13, 22, 35, 57, 92, 149, 241, 390$. Since $390 \equiv 6 \pmod{24}$, the first m is $\boxed{10}$.

10. Let p, q, and r be the roots of the polynomial $f(t) = t^3-2022t^2+2022t-337$. Given

$$\begin{aligned}
x &= (q-1)\left(\frac{2022-q}{r-1}+\frac{2022-r}{p-1}\right)\\
y &= (r-1)\left(\frac{2022-r}{p-1}+\frac{2022-p}{q-1}\right)\\
z &= (p-1)\left(\frac{2022-p}{q-1}+\frac{2022-q}{r-1}\right)
\end{aligned}$$

compute $xyz-qrx-rpy-pqz$.

Answer: $\mathbf{-674}$

Solution: What is unsatisfying about this is the nonhomogeneity of $p-1$, and the key realization to fixing this is to note that the quadratic and linear coefficients in the polynomial are identical, i.e. $qr+rp+pq = p+q+r \Rightarrow p(q-1)+q(r-1)+r(p-1) = 0$. Therefore, we now can denote with dimensionless variables a, b, c, $p(q-1)$, $q(r-1)$, and $r(p-1)$ respectively, refactoring the equations into

$$\begin{aligned}
px &= a\left(\frac{pq+qr}{b}+\frac{qr+rp}{c}\right) = -(rp+pq)+aS\\
qy &= b\left(\frac{qr+rp}{c}+\frac{rp+pq}{a}\right) = -(pq+qr)+bS\\
rz &= c\left(\frac{rp+pq}{a}+\frac{pq+qr}{b}\right) = -(qr+rp)+cS
\end{aligned}$$

where $a+b+c=0$ and $S=\frac{rp+pq}{a}+\frac{pq+qr}{b}+\frac{qr+rp}{c}$. If we denote $i=qr, j=rp, k=pq$, and $x'=-px=(j+k)-aS$ with y' and z' similarly, then we can rewrite the target equation as $-\frac{1}{pqr}(x'y'z'-(i^2x'+j^2y'+k^2z'))$.

Now, note that if we let $g(x,y,z)=xyz-(i^2x+j^2y+k^2z)$, then we can see that $g(j+k,k+i,i+j)=2ijk$. Then we have

$$\begin{aligned}
&g(x',y',z')-g(j+k,k+i,i+j)\\
=&(j+k-aS)(k+i-bS)(i+j-cS)-(j+k)(k+i)(i+j)+\sum_{cyc} i^2aS\\
=&-\sum_{cyc}(k+i)(i+j)aS+\sum_{cyc}(j+k)bcS^2-abcS^3+\sum_{cyc} i^2aS\\
=&abcS^3\left(\frac{1}{S}\sum_{cyc}\frac{j+k}{a}-1\right)\\
=&0
\end{aligned}$$

meaning we have $g(x',y',z')=g(j+k,k+i,i+j)=2ijk$. Finally, we have

$$\begin{aligned}
xyz-(qrx+rpy+pqz) &= -\frac{1}{pqr}(x'y'z'-(i^2x'+j^2y'+k^2z'))\\
&= -\frac{1}{pqr}g(x',y',z')\\
&= -\frac{1}{pqr}\cdot 2ijk\\
&= -2pqr\\
&= \boxed{-674}.
\end{aligned}$$