Time limit: 60 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. Let x be a real number such that $x^2 - x + 1 = 7$ and $x^2 + x + 1 = 13$. Compute the value of x^4.

2. Let f and g be linear functions such that $f(g(2021)) - g(f(2021)) = 20$. Compute $f(g(2022)) - g(f(2022))$. (Note: A function h is linear if $h(x) = ax + b$ for all real numbers x.)

3. Let x be a solution to the equation $\lfloor x \lfloor x + 2 \rfloor + 2 \rfloor = 10$. Compute the smallest C such that for any solution x, $x < C$. Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor -4.25 \rfloor = -5$.

4. Let θ be a real number such that $1 + \sin 2\theta - \left(\frac{1}{2}\sin 2\theta\right)^2 = 0$. Compute the maximum value of $(1 + \sin\theta)(1 + \cos\theta)$.

5. Compute the sum of the real solutions to $\lfloor x \rfloor \{x\} = 2020x$. Here, $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x, and $\{x\} = x - \lfloor x \rfloor$.

6. Let f be a real function such that for all $x \neq 0, x \neq 1$,

$$f(x) + f\left(-\frac{1}{x-1}\right) = \frac{9}{4x^2} + f\left(1 - \frac{1}{x}\right).$$

   Compute $f\left(\frac{1}{2}\right)$.

7. Let $z_1, z_2, \ldots, z_{2020}$ be the roots of the polynomial $z^{2020} + z^{2019} + \cdots + z + 1$. Compute

$$\sum_{i=1}^{2020} \frac{1}{1 - z_i^{2020}}.$$

8. Let $f(w) = w^3 - rw^2 + sw - \frac{4\sqrt{2}}{27}$ denote a polynomial, where $r^2 = \left(\frac{8\sqrt{2}+10}{7}\right)s$. The roots of f correspond to the sides of a right triangle. Compute the smallest possible area of this triangle.

9. Compute the sum of the positive integers $n \leq 100$ for which the polynomial $x^n + x + 1$ can be written as the product of at least 2 polynomials of positive degree with integer coefficients.

10. Given a positive integer n, define $f_n(x)$ to be the number of square-free positive integers k such that $kx \leq n$. Then, define $v(n)$ as

$$v(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_n\left(i^2\right) - 6f_n(ij) + f_n\left(j^2\right).$$

    Compute the largest positive integer $2 \leq n \leq 100$ for which $v(n) - v(n-1)$ is negative. (Note: A square-free positive integer is a positive integer that is not divisible by the square of any prime.)