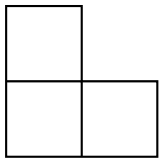
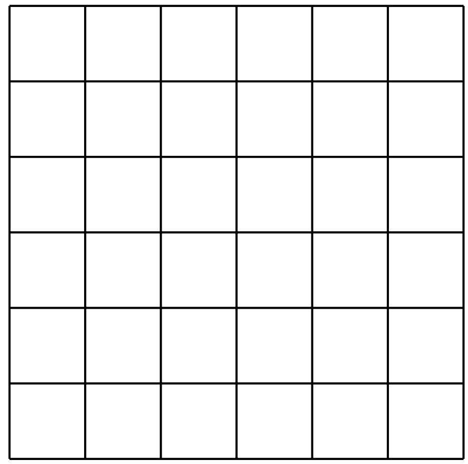
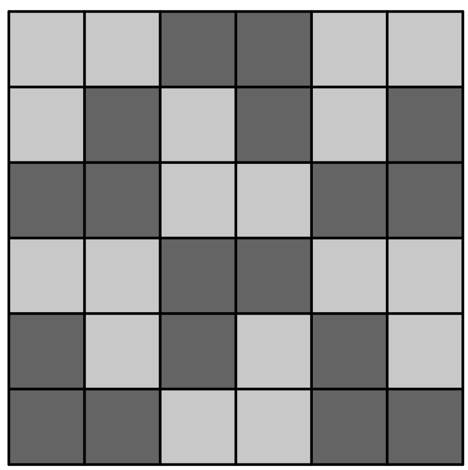
1. What is the maximum number of L-shaped pieces (the piece on the left) that can fit in the 6×6 grid shown below? You may assume that you can rotate the L-shaped pieces.





Answer: 12

Solution: Rotating the "L" allows us to find a way to fit the piece into the grid a total of $\boxed{12}$ times. Here is one such configuration.



2. Let N_1 be the answer to question 1.

Suppose $N_1 + 1 + \frac{1}{N_1}$ can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute a.

Answer: 157

Solution: Note $\frac{a}{b} = \frac{N_1^2+N_1+1}{N_1}$. By the Euclidean algorithm, $N_1^2 + N_1 + 1$ and N_1 are relatively prime, so $a = N_1^2 + N_1 + 1$. By question 1, we have $N_1 = 12$, so $a = 12^2 + 12 + 1 = \boxed{157}$.

3. Every day after school, Alice drops 1 marble on the ground, Bob drops 2 marbles on the ground, and Carol drops 3 marbles on the ground. After some number of days, there are a total of 24 marbles on the ground. How many of these marbles were dropped by Bob?

Answer: 8

Solution: There are a total of $1 + 2 + 3 = 6$ marbles being dropped on the ground at a time. Therefore, if a total of 24 marbles are on the ground, then the action of throwing marbles on the ground happened for 4 days. Therefore, Bob dropped a total of $2 \cdot 4 = \boxed{8}$ marbles.

4. Let N_3 be the answer to question 3.

The hype-house for Beta Mu Tau has N_3 doors, each numbered 1 through N_3. Jingyuan enters through a door and exits through a door (which doesn't have to be a different door from the first), such that the sum of the numbers on the two doors is even. Compute the number of possible ways Jingyuan could enter and exit the hype-house.

Answer: 32

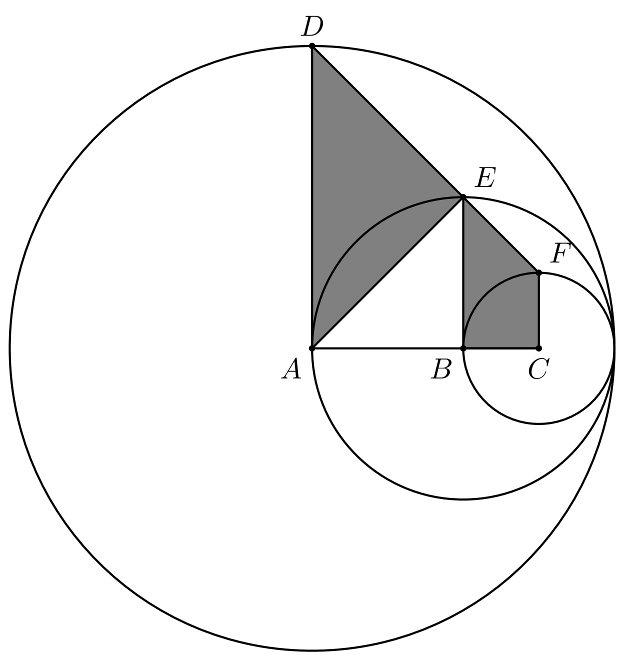
Solution: There are $\lfloor N_3/2 \rfloor$ even-numbered doors and $\lfloor (N_3+1)/2 \rfloor$ odd-numbered doors. If Jingyuan enters through an even-numbered door, then Jingyuan must exit from any of the even-numbered doors. Analogously, if Jingyuan enters through an odd-numbered door, then Jingyuan must exit from any of the odd-numbered doors. In total, there are

$$\left\lfloor \frac{N_3}{2} \right\rfloor^2 + \left\lfloor \frac{N_3+1}{2} \right\rfloor^2$$

ways for Jingyuan to enter and exit. Plugging in $N_3 = 8$ yields an answer of $4^2 + 4^2 = \boxed{32}$.

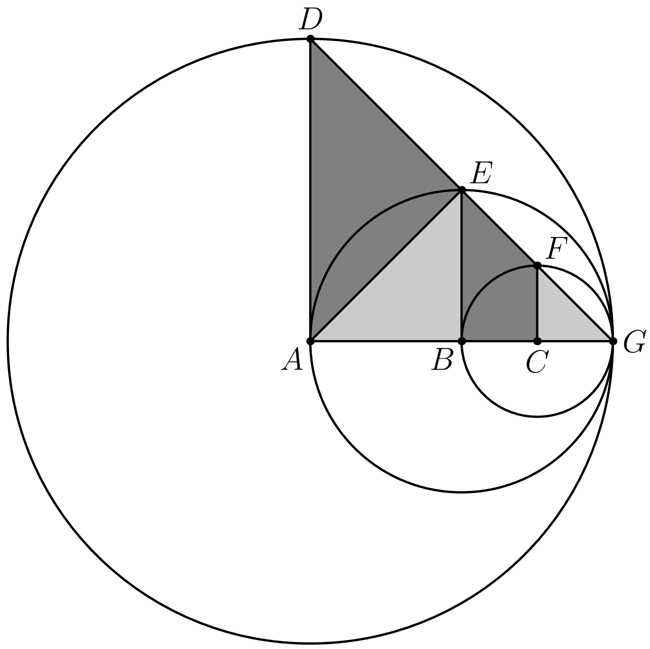
5. Let N_4 be the answer to question 4.

Three circles with centers A, B, and C are internally tangent to each other at a common point, as shown in the diagram below, and have radii N_4, $\frac{N_4}{2}$, and $\frac{N_4}{4}$, respectively. Points D, E, and F lie on the circles centered at A, B, and C, respectively, such that angles $\angle DAB = \angle EBC = \angle FCB = 90°$. What is the area of the shaded figure?



Answer: 352

Solution: Consider the following diagram.



We can take the area of the big 45-45-90 triangle $\triangle ADG$ with legs of length N_4, then subtract both the area of the 45-45-90 triangle $\triangle ABE$ with legs of length $\frac{N_4}{2}$ and the area of the 45-45-90 triangle $\triangle CFG$ with legs of length $\frac{N_4}{4}$. The area of $\triangle ADG$ is $\frac{N_4^2}{2}$, and the areas of $\triangle ABE$ and $\triangle CFG$ are $\frac{N_4^2}{8}$ and $\frac{N_4^2}{32}$, respectively. Therefore, the shaded area is

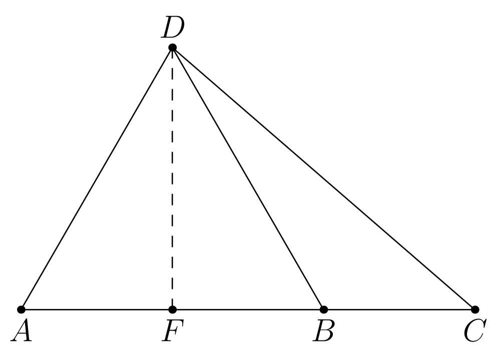
$$\frac{N_4^2}{2} - \frac{N_4^2}{8} - \frac{N_4^2}{32} = \frac{11}{32}N_4^2.$$

Plugging in $N_4 = 32$ gives us our answer of $\boxed{352}$.

6. Points A, B, and C lie on a line, in that order, from left to right. Point D is plotted such that triangle $\triangle ADB$ is equilateral. If $BC = 2$ and $DC = 2\sqrt{13}$, compute AC.

Answer: 8

Solution: We will build the following diagram.



Let the altitude from D intersect $\overline{AB}$ at F; then let $FB = x$ so that $DF = x\sqrt{3}$. Thus, by the Pythagorean Theorem,

$$(2+x)^2 + 3x^2 = FC^2 + FD^2 = CD^2 = \left(2\sqrt{13}\right)^2 = 52.$$

This rearranges to $0 = 4x^2 + 4x - 48 = 4(x+4)(x-3)$, which has positive solution $x = 3$. Hence, $AB = 2x = 6$, so $AC = AB + BC = 6 + 2 = \boxed{8}$.

7. Let N_6 be the answer to question 6.

   Shreyas plays a game with N_6 buckets. During a round, he chooses two buckets uniformly at random to keep track of, then drops a ball into one of the N_6 buckets uniformly at random. If the ball lands in one of the chosen buckets, the game ends. Otherwise, he removes the ball and the bucket that the ball landed in, and plays another round. Compute the probability that this game will continue until Shreyas has two buckets left.

   Answer: $\mathbf{\frac{1}{28}}$

   Solution: If there are currently $n > 2$ buckets, the probability that we do not end on this round is $\frac{n-2}{n}$. Thus, the desired probability is

$$\frac{N_6 - 2}{N_6} \cdot \frac{N_6 - 3}{N_6 - 1} \cdot \frac{N_6 - 4}{N_6 - 2} \cdots \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{2}{N_6 \cdot (N_6 - 1)}.$$

   Plugging in our value for N_6 gives us $\frac{2}{8 \cdot 7} = \boxed{\frac{1}{28}}$.

8. The answer to question 7 can be written in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Let N_7 be $a + b$.

   There are N_7 red dots and some number of blue dots connected by line segments. Each red dot is connected to exactly 4 blue dots, and each blue dot is connected to exactly 2 red dots. Furthermore, each blue dot is connected to exactly 2 other blue dots, and no two red dots are connected. Compute the number of line segments drawn.

   Answer: 174

   Solution: Let $r = N_7$ and b be the number of red and blue dots, respectively. To begin, we claim that $b = 2r$. For this, we count the number of line segments connecting red and blue dots. On one hand, each red dot is connected to four blue ones, so the number of line segments connecting red and blue dots is $4r$. On the other hand, each blue dot is connected to two red ones, so this value is also equal to $2b$. It follows that

$$4r = 2b,$$

   so $b = 2r$.

   To complete the problem, we count the total number of line segments in cases.

   - There are no line segments connecting a red dot to a red dot.
   - As above, there are $4r$ line segments connecting a red dot to a blue dot.
   - Lastly, each blue dot is connected to exactly 2 other blue dots. Summing over all blue dots, this would produce $2b$ line segments, but each of these line segments has been double-counted because both endpoints are blue dots. Thus, there are b line segments connecting a blue dot to a blue dot.

   Totaling, we have $4r + b = 6r = 6N_7 = 6 \cdot 29 = \boxed{174}$ line segments.

9. Let N_8 be the **sum of the digits** of the answer to question 8.

   For each positive integer k, define the integers

$$a_k = k + 2k + \cdots + (k-1)k + k \cdot k$$

   and

$$b_k = (k)(2k)\cdots((k-1)k)(k \cdot k).$$

   What is the largest positive integer k less than or equal to N_8 such that b_k is NOT divisible by a_k?

   Answer: 12

   Solution: Note $b_k = k^k k!$ and

$$a_k = k + 2k + 3k + \cdots + k^2 = k(1+2+3+\cdots+k) = \frac{k^2(k+1)}{2}.$$

   Therefore, we want to investigate when

$$x_k = \frac{b_k}{a_k} = \frac{2k^{k-2}k!}{k+1}.$$

   is not an integer. We claim that x_k is not an integer if and only if $k > 1$ and $k+1$ is prime. Quickly, we note that x_k is an integer for $k = 1$, which explains why we require $k > 1$. Otherwise, we may assume that $k \geq 2$ so that k^{k-2} is an integer in the argument which follows.

   In one direction, if $k+1$ is prime, then $k+1$ divides no factor in the numerator of x_k, so x_k is not an integer. Conversely, suppose $k+1$ is composite. We have the following cases.

   - If $k+1$ can be factored as $k+1 = ab$ with $1 < a, b < k+1$ and $a \neq b$, then we see $k+1 = ab \mid k!$. Thus, x_k is an integer. This case applies if $k+1$ is a multiple of distinct primes or if $k+1 = p^n$ is a power of a prime where $n > 2$.
   - Lastly, suppose $k+1 = p^2$ is the square of a prime. If $k = 3$, then $x_k = 2 \cdot 3 \cdot 6/4 = 9$ is an integer. Otherwise, $p > 2$, so $k!$ contains the factors p and $2p < p^2$, so $k+1$ again divides $k!$, meaning that x_k is an integer.

   Thus, x_k indeed fails to be an integer if and only if $k > 1$ and $k+1$ is prime. With $N_8 = 12$, we see that $k = \boxed{12}$ is the largest such integer $k \leq 12$.

10. Albert has two cups, A and B, containing a and b liters of water, respectively. Albert pours one-third of cup A's contents into cup B, and then pours one-half of cup B's contents into cup A. He notices that cups A and B now contain b and a liters of water, respectively. If the water in both cups combines for a total of 24 liters, compute a.

    Answer: 9

    Solution: After the first pour, cups A and B contain $\frac{2a}{3}$ and $\frac{a}{3} + b$ liters, respectively. After the second pour, cups A and B contain $\frac{5a}{6} + \frac{b}{2}$ and $\frac{a}{6} + \frac{b}{2}$ liters, respectively. Thus,

$$\frac{5a}{6} + \frac{b}{2} = b,$$

    so $a = \frac{3}{5} \cdot b$. We also know that $a + b = 24$, so plugging in $a = \frac{3}{5} \cdot b$ into that equation gives us $a = \boxed{9}$.

11. Compute the 4-digit perfect square of the form $\underline{AABB}$, where A and B are digits and $A \neq 0$.

Answer: 7744

Solution: Let $n^2 = \underline{AABB} = 11(100A + B)$. Then $11 \mid n^2$ implies $11 \mid n$, so writing $n = 11k$ implies that $100A + B = 11k^2$ for some integer k. Since $100 \leq 100A + B < 1000$, it follows that $10 \leq k^2 \leq 90$, so $k \in \{4, 5, \dots, 9\}$. We now tabulate as follows.

k	4	5	6	7	8	9
$11k^2$	176	275	396	539	704	891

The only $11k^2$ expressible as $100A + B$ is 704, and $n^2 = 11 \cdot 704 = \boxed{7744}$.

12. The value 56 is written in its binary form as 111000. Jingyuan takes all rearrangements of these ones and zeroes (including 111000) and sets the leftmost digit of each rearrangement equal to 0 (regardless if the digit was a 1 or 0 before). He then sums up all the resulting binary values, and then converts this sum to base 10. Compute this sum.

Answer: 310

Solution: We know that the leading term will be made 0, so there are two possible cases.

- There was a 1 in the leading term. This means that the remaining 5 bits consist of 3 zeroes and 2 ones. The $\binom{5}{2} = 10$ possible combinations are therefore as follows.

  $$\begin{array}{c} 11000 \\ 10100 \\ 10010 \\ \vdots \end{array}$$

  If we sum up all these values, we see that there are 4 ones in each place value, so our sum is $4(1 + 2 + 4 + 8 + 16) = 124$.

- There was a 0 in the leading term. This means that the remaining 5 bits consist of 3 ones and 2 zeroes. The $\binom{5}{2} = 10$ possible combinations are therefore as follows.

  $$\begin{array}{c} 11100 \\ 11010 \\ 11001 \\ \vdots \end{array}$$

  If we sum up all these values, we see that there are 6 ones in each place value, so our sum is $6(1 + 2 + 4 + 8 + 16) = 186$.

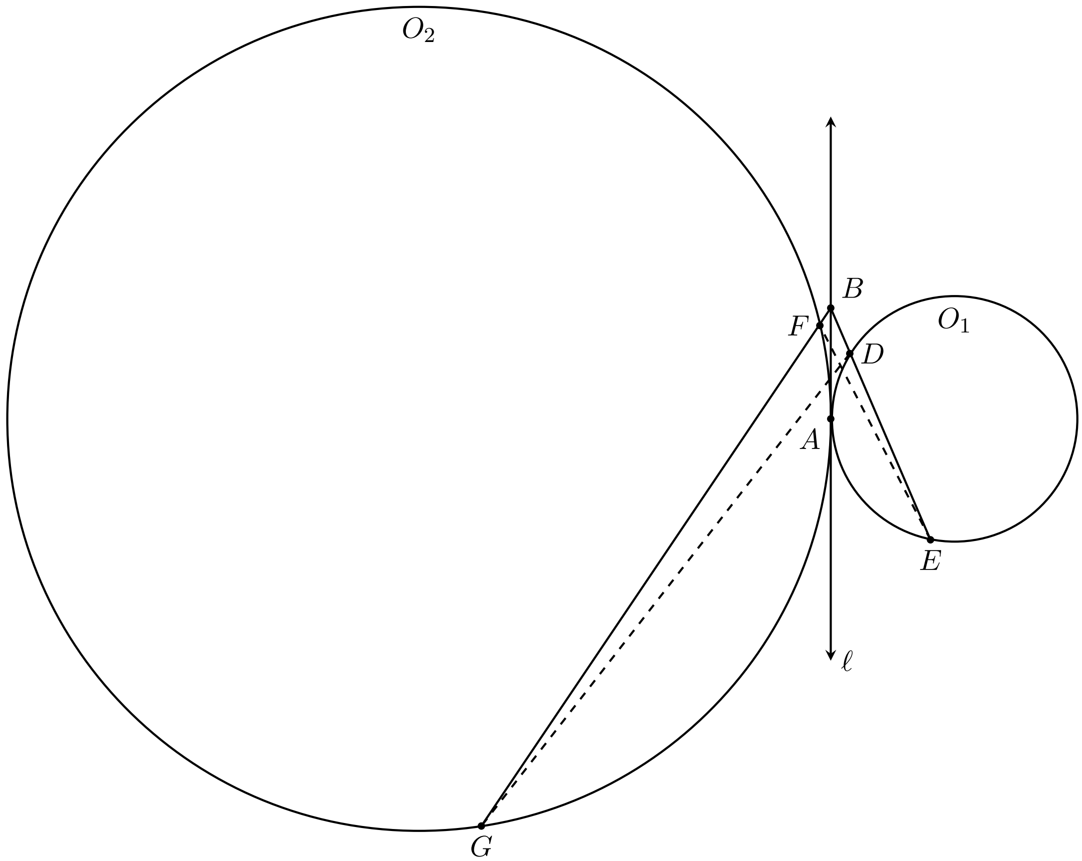
Therefore, the sum of all values is $124 + 186 = \boxed{310}$.

13. Let N_{10} be the answer to question 10, N_{11} be the **sum of the digits** of the answer to question 11, and N_{12} be the **sum of the digits** of the answer to question 12.

Two circles O_1 and O_2 are externally tangent to line ℓ at the same point A, and the two circles are externally tangent to each other. Point B lies on line ℓ such that $AB = N_{11}$. There is a line through point B that intersects circle O_1 at points D and E, where $BD < BE$ and $BD = N_{10}$, and another line through point B that intersects circle O_2 at points F and G, where $BF < BG$ and $BF = N_{12}$. Compute $\frac{DG}{EF}$.

Answer: $\frac{9}{4}$

Solution: We have the following diagram.



Using Power of a Point, we see

$$BD \cdot BE = BA^2 = BF \cdot BG.$$

Now, because $\frac{BF}{BE} = \frac{BD}{BG}$ and $\angle FBE = \angle DBG$, we can say $\triangle BEF \cong \triangle BGD$ by SAS similarity. Therefore, $\frac{DG}{EF} = \frac{BD}{BF} = \frac{N_{10}}{N_{12}} = \boxed{\frac{9}{4}}$.

14. Anton has two standard six-sided dice where opposite faces always add up to 7. He puts the first die on a table, and stacks the second die on top of the first die so that there are 9 visible faces and 3 non-visible faces. The product of the values of the non-visible faces does not divide the product of the values of the visible faces. What is the sum of the non-visible faces?

Answer: 12

Solution: We work with the two dice individually since it is necessary that, for at least one of the two die, the non-visible faces do not divide the visible faces.

For the bottom die, we determine via simple casework that the only possibility is if the two hidden faces are 2 and 5. For the top die, we similarly determine that the only possibility is if the hidden face is 5. In general, only the location of the 5's matter, as the other factors of the visible faces will always be in excess relative to the factors of the hidden faces.

Now, we show that both of these conditions must hold instead of at least one. If the bottom die has 2 and 5 hidden and the top die has 5 visible, then the factors of 5 end up neutralizing each other. Similarly, if the top die has 5 hidden and the bottom die has 5 (and 2) visible, the factors of 5 also neutralize each other. All other factors are in excess for the visible faces, so divisibility occurs.

Thus, the only possibility is if the hidden faces are 2 and 5 for the bottom die and 5 for the top die, so our answer is $2 + 5 + 5 = \boxed{12}$.

15. Let N_{14} be the answer to question 14.

Compute the area in the xy-plane above the line $y = 0$ and to the left of the line $x = N_{14}$, but below the graph of $y = N_{14}\lfloor x \rfloor$. (Here, $\lfloor m \rfloor$ is defined as the greatest integer less than or equal to m. For example, $\lfloor 3.14 \rfloor = 3$ and $\lfloor 4 \rfloor = 4$.)

Answer: 792

Solution: We add a new line $y = N_{14}x$. When graphing the lines, we see that the answer can be found by computing the area bounded by the lines $y = 0$, $y = N_{14}x$, and $x = N_{14}$, and then subtracting the area of the N_{14} triangles bounded between $y = N_{14}x$ and $y = N_{14}\lfloor x \rfloor$. Therefore, our answer is

$$\frac{N_{14} \cdot N_{14}^2}{2} - N_{14} \cdot \frac{1 \cdot N_{14}}{2} = \frac{N_{14}^2(N_{14} - 1)}{2}.$$

Plugging in $N_{14} = 12$ gives $\boxed{792}$.

16. Let N_{14} be the answer to question 14.

How many sequences of distinct positive integers starting with 1 and ending with $2^{N_{14}}$ are there such that each integer of the sequence (except for 1) is divisible by the previous integer?

Answer: 2048

Solution: We can essentially think of this sequence as starting from 1, and then multiplying by combinations of the factors of $2^{N_{14}}$ until we get to exactly $2^{N_{14}}$. In order to count all these different combinations of factors, we can think of the sequence as

$$2 \circ 2 \circ 2 \circ 2 \circ \cdots \circ 2$$

where the $\circ$ indicates whether or not we partition $2^{N_{14}}$ at this spot; each choice of partitions corresponds to a unique sequence. For example, if we partition only at the first and last $\circ$, this corresponds to the sequence $1, 2, 2^{N_{14}-1}, 2^{N_{14}}$.

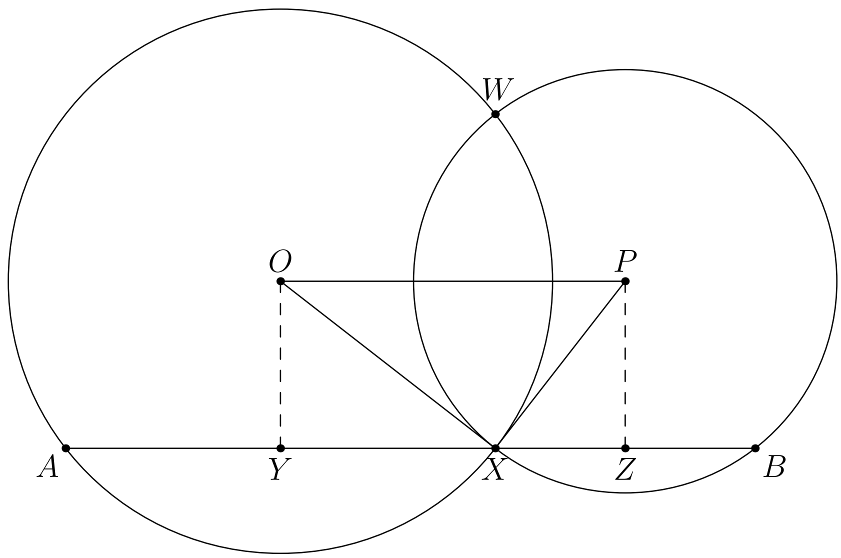
There are $N_{14} - 1$ possible spots to add a partition, and a partition is either added or not added at that spot. Thus, there are a total of $2^{N_{14}-1}$ possible partition combinations, which also means there are $2^{N_{14}-1} = \boxed{2048}$ sequences.

17. Let N_{15} be the **sum of the digits** of the answer to question 15 and N_{16} be the **sum of the digits** of the answer to question 16.

Consider two circles D_O and D_P centered at points O and P with radii N_{15} and N_{16} respectively, and suppose they intersect at points W and X with $\overline{OX} \perp \overline{PX}$. Let ℓ be a line passing through X which intersects circle D_O elsewhere at A and circle D_P elsewhere at B. Compute the maximum possible value of AB.

Answer: $4\sqrt{130}$

Solution: We will build the following diagram.



Let Y and Z be the projections of O and P onto ℓ; i.e., Y and Z lie on ℓ while $\overline{OY} \perp \overline{AB}$ and $\overline{PZ} \perp \overline{AB}$. Note $AB = 2YZ$ because

$$YZ = YX + XZ = \frac{1}{2}AX + \frac{1}{2}XB,$$

so it is equivalent to maximize YZ.

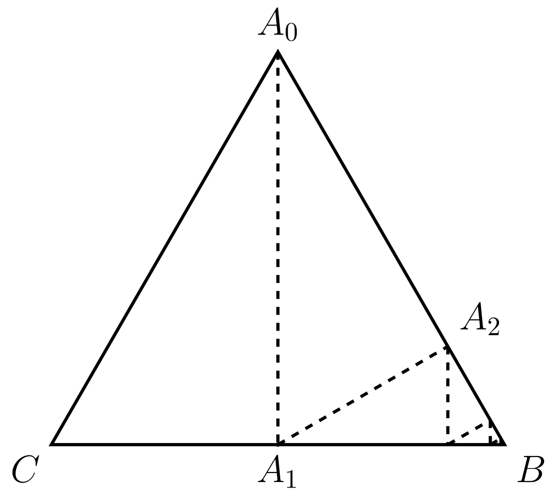
Fixing X on ℓ, we can imagine rotating triangle $\triangle OXP$ around X to maximize the length of $\overline{YZ}$, which is the projection of the segment $\overline{OP}$ onto ℓ. This projection is maximized precisely when $\overline{OP} \parallel \ell$, where we get $YZ = OP$. By the Pythagorean theorem, $OP = \sqrt{N_{15}^2 + N_{16}^2}$, so we get

$$AB = 2OP = 2\sqrt{N_{15}^2 + N_{16}^2}.$$

Plugging in $N_{15} = 18$ and $N_{16} = 14$ yields $AB = \boxed{4\sqrt{130}}$.

18. Let N_{18} be the answer to this question.

    The equilateral triangle $\triangle A_0BC$, as shown in the diagram below, has side length $\sqrt{3}(N_{18} - 2)$. Let the altitude from A_0 intersect $\overline{BC}$ at A_1. Then for each positive integer i, an altitude from A_i intersects $\overline{A_{i-1}B}$ at A_{i+1}. This process is continued indefinitely, and points A_0, A_1, and A_2 are shown in the diagram. Compute the sum of the lengths $A_0A_1 + A_1A_2 + A_2A_3 + \cdots$ (these lengths are represented by the dashed lines in the diagram).



Answer: 3

Solution: Note that $A_{i+1}A_{i+2} = \frac{1}{2}A_iA_{i+1}$ for each $i \geq 0$, so we have an infinite series when we sum up these lengths. Namely, if we say $s = BC$ is the side length of $\triangle ABC$, then the sum is

$$\frac{\sqrt{3}}{2}s + \frac{\sqrt{3}}{4}s + \frac{\sqrt{3}}{8}s + \cdots = \sqrt{3} \cdot s.$$

Plugging in our value for s and equating it to N_{18} gives us our answer of $\sqrt{3} \cdot \sqrt{3}(N_{18} - 2) = N_{18}$, so $N_{18} = 3$.

19. Let N_{20} be the answer to question 20.

Leanne builds a pyramid out of equally sized blocks. The base has N_{20} blocks, and each successive layer has 2 fewer blocks. If there is exactly 1 block on the top level of the pyramid, how many blocks are in the pyramid?

Answer: 676

Solution: The total number of blocks is the sum of the first m consecutive odd numbers for some integer m. To find m, we can use the fact that $2m - 1 = N_{20}$, so

$$m = \frac{N_{20} + 1}{2}.$$

It is known that the sum of the first m consecutive odd numbers is m^2. Therefore, our answer is $\left(\frac{N_{20}+1}{2}\right)^2$. Plugging in $N_{20} = 51$ from question 20 gives us 26^2, so $\boxed{676}$.

20. Let N_{19} be the answer to question 19.

The integer $N_{19} - 1$ can be written in the form p^2q^3, where p and q are distinct one-digit prime numbers. Compute $10p + q - 2$.

Answer: 51

Solution: After doing question 19, we conclude that the answer to question 19 is a perfect square, so we can say that $N_{19} = x^2$ for some integer x. Thus,

$$p^2q^3 = N_{19} - 1 = x^2 - 1 = (x - 1)(x + 1).$$

Suppose N_{19} was odd, and thus $x^2 - 1 = (x-1)(x+1)$ is even. Therefore, $8|x^2 - 1$, and thus $q = 2$. However, this means $10p + q - 2$ is even, which causes the pyramid in N_{19} to not have 1 block at the top level. This means that N_{19} must be even, and $x - 1$ and $x + 1$ are both odd, meaning they are relatively prime. Thus, we know one of $x - 1$ or $x + 1$ must be p^2 and one must be q^3.

Our problem now reduces down to finding a perfect square of a one-digit prime and a perfect cube of a different one-digit prime that are exactly 2 away from each other. After checking the possible one-digit primes, we notice that 25 (which is the square of a one-digit prime) is exactly 2 away from 27 (which is the cube of a one-digit prime). Therefore, $p = 5$ and $q = 3$, so $10p + q - 2 = \boxed{51}$.