1. Given that $7 \times 22 \times 13 = 2002$, compute $14 \times 11 \times 39$.

   Answer: 6006

   Solution: Note that $7 \times 22 = 14 \times 11$ and $13 \times 3 = 39$. Therefore, $14 \times 11 \times 39 = 3 \times (7 \times 22 \times 13) = 3 \times 2002 = \boxed{6006}$.

2. Ariel the Frog is on the top left square of a 8×10 grid of squares. Ariel can jump from any square on the grid to any adjacent square, including diagonally adjacent squares. What is the minimum number of jumps required for Ariel to reach the bottom right corner?

   Answer: 9

   Solution: If Ariel wants to reach the bottom corner with the minimum number of jumps, she should jump diagonally each time she has the opportunity to. She can do so until the very last row, where the bottom right square is 2 jumps away. It takes her 7 jumps to jump from the top left square to the bottom row (diagonally) and 2 more jumps to get to the destination. Therefore, it takes her $7 + 2 = \boxed{9}$ total jumps.

3. The distance between two floors in a building is the vertical distance from the bottom of one floor to the bottom of the other. In Evans hall, the distance from floor 7 to floor 5 is 30 meters. There are 12 floors on Evans hall and the distance between any two consecutive floors is the same. What is the distance, in meters, from the first floor of Evans hall to the 12th floor of Evans hall?

   Answer: 165

   Solution: Since the distance from the 7th floor to the 5th floor is 30 meters, the distance between any two consecutive floors is 15 meters. It takes 11 floors to get from the 1st floor to the 12th floor, so the total distance from the 1st floor to the 12th floor is $11 \times 15 = \boxed{165}$ meters.

4. A circle of nonzero radius r has a circumference numerically equal to $\frac{1}{3}$ of its area. What is its area?

   Answer: 36π

   Solution: The area and circumference of a circle of radius r are πr^2 and $2\pi r$, respectively. Thus, we obtain the equation $\frac{\pi}{3} r^2 = 2\pi r$, so $r = 6$. The area of the circle, therefore, is $\boxed{36\pi}$.

5. As an afternoon activity, Emilia will either play exactly two of four games (TwoWeeks, DigBuild, BelowSaga, and FlameSymbol) or work on homework for exactly one of three classes (CS61A, Math 1B, Anthro 3AC). How many choices of afternoon activities does Emilia have?

   Answer: 9

   Solution: There are $\binom{4}{2} = 6$ choices of games, and 3 choices of homework, so in total there are $6 + 3 = \boxed{9}$ choices of afternoon activities.

6. Matthew wants to buy merchandise of his favorite show, Fortune Concave Decagon. He wants to buy figurines of the characters in the show, but he only has 30 dollars to spend. If he can buy 2 figurines for 4 dollars and 5 figurines for 8 dollars, what is the maximum number of figurines that Matthew can buy?

   Answer: 17

   Solution: Note Matthew can buy at most 5 figurines with every 8 dollars, so the maximum number of figurines that he can get is $5 \times 3 + 2 = \boxed{17}$.

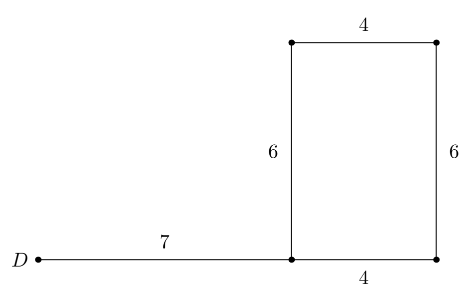
7. When Dylan is one mile from his house, a robber steals his wallet and starts to ride his motorcycle from Dylan's current position in the direction opposite from Dylan's house at 40 miles per hour. Dylan dashes home at 10 miles per hour and, upon reaching his house, begins driving his car at 60 miles per hour in the direction of the robber's motorcycle. How long, starting from when the robber steals the wallet, does it take for Dylan to catch the robber? Express your answer in minutes.

   Answer: 21

   Solution: It takes Dylan $\frac{1}{10}$ of an hour to get back to his house. In that time, the robber has traveled $40 \cdot \frac{1}{10} = 4$ miles away, and since Dylan's original position is 1 mile from his house, the robber is $4 + 1 = 5$ miles away from Dylan's house. Dylan's relative speed to the robber is $60 - 40 = 20$ miles per hour, so it will take $\frac{5}{20} = \frac{1}{4}$ hours to reach the robber. Thus, it takes Dylan $60 \cdot \left(\frac{1}{4} + \frac{1}{10}\right) = \boxed{21}$ minutes to catch the robber.

   Note: If one is not familiar with relative speed, consider the following. After Dylan starts driving, let t be the amount of time for Dylan to catch the robber. Using the formula of distance equals rate times time and the fact that at time t, the robber is currently $5+40t$ miles away from Dylans house, we find that $5 + 40t = 60t$, which rearranges to $5 = 20t$. We can thus proceed as in the solution.

8. Deepak the Dog is tied with a leash of 7 meters to a corner of his 4 meter by 6 meter rectangular shed such that Deepak is outside the shed. Deepak cannot go inside the shed, and the leash cannot go through the shed. Compute the area of the region that Deepak can travel to.



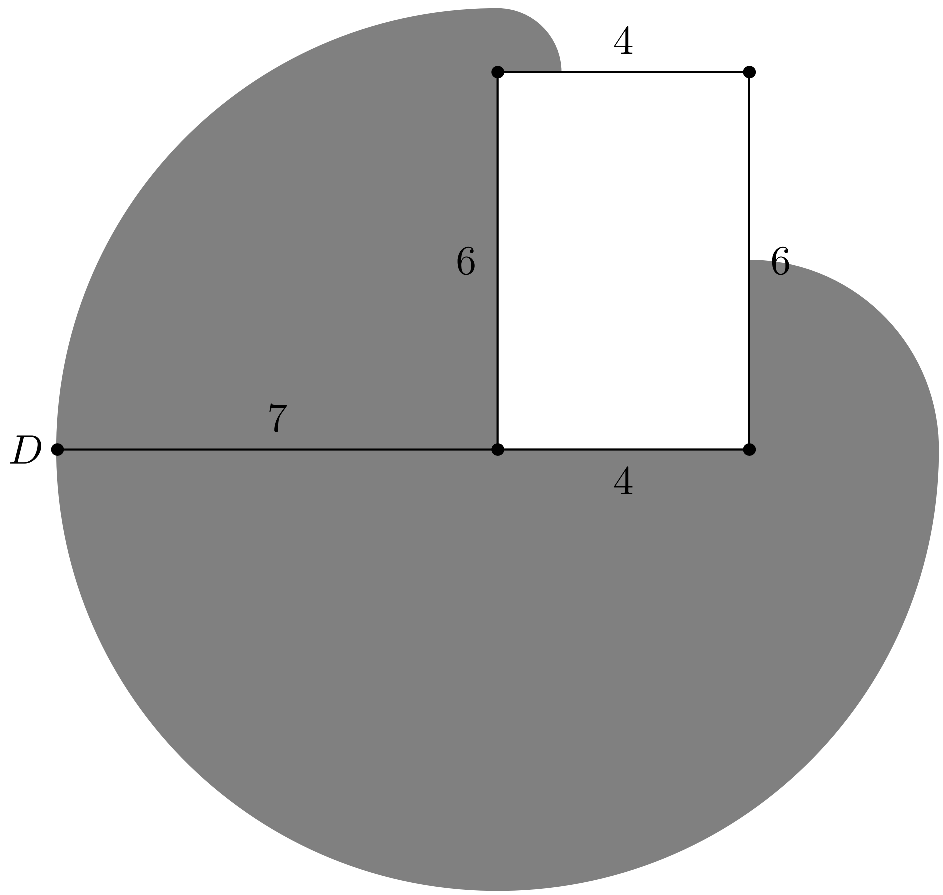
   Answer: $\mathbf{\frac{157\pi}{4}}$ **OR** $\mathbf{39.25\pi}$ **OR** $\mathbf{39\frac{1}{4}\pi}$

   Solution: The desired region is indicated as the shaded part in the graph above. The total area is $\frac{1}{4}\pi \cdot 1^2 + \frac{1}{4}\pi \cdot 3^2 + \frac{3}{4}\pi \cdot 7^2 = \boxed{\frac{157\pi}{4}}$.

9. The quadratic equation $a^2x^2 + 2ax - 3 = 0$ has two solutions for x that differ by a, where $a > 0$. What is the value of a?

   Answer: 2

   Solution: The above equation can be factored as $(ax + 3)(ax - 1) = 0$, so $x = -\frac{3}{a}$ or $x = \frac{1}{a}$ as $a \neq 0$. The difference in the solutions is then $\frac{4}{a} = a$. Given $a > 0$, $a = \boxed{2}$.



10. Find the number of ways to color a 2×2 grid of squares with 4 colors such that no two (non-diagonally) adjacent squares have the same color. Each square should be colored entirely with one color. Colorings that are rotations or reflections of each other should be considered different.

Answer: 84

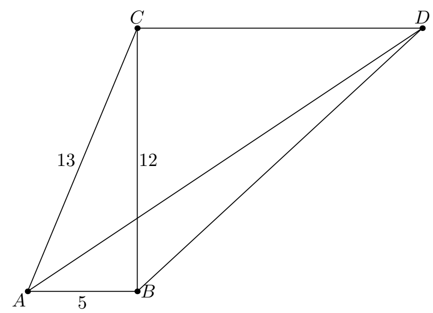
Solution: Consider the two squares in the top-right and the bottom-left of the grid. If these two squares have the same color, then we have 4 possibilities for the color of these two squares and $3 \cdot 3$ possibilities for the color of the other two squares. If the top-right and the bottom-left squares have different colors, then we have $4 \cdot 3$ ways to pick these two squares and $2 \cdot 2$ ways to pick the other two squares. Thus, the total number of ways is $4 \cdot 3 \cdot 3 + 4 \cdot 3 \cdot 2 \cdot 2 = \boxed{84}$.

11. Given that $\frac{1}{y^2+5} - \frac{3}{y^4-39} = 0$, and $y \geq 0$, compute y.

Answer: 3

Solution: Substitute $a = y^2$ to get $\dfrac{1}{a+5} = \dfrac{3}{a^2-39}$, so $3a+15 = a^2-39$. Then $a^2-3a-54 = 0$, so $(a-9)(a+6) = 0$. Since $y^2 \geq 0$, $a \geq 0$, and $a = 9$. Then $y = \boxed{3}$.

12. Right triangle ABC has $AB = 5$, $BC = 12$, and $CA = 13$. Point D lies on the angle bisector of $\angle BAC$ such that CD is parallel to AB. Compute the length of BD.



Answer: $\sqrt{313}$

Solution: Notice that because CD is parallel to AB, $\angle CDA = \angle BAD = \angle DAC$. Thus, $AC = CD = 13$, so $BC = \sqrt{BC^2 + CD^2} = \sqrt{12^2 + 13^2} = \boxed{\sqrt{313}.}$

13. Let x and y be real numbers such that $xy = 4$ and $x^2y + xy^2 = 25$. Find the value of $x^3y + x^2y^2 + xy^3$.

Answer: $\frac{561}{4}$ **OR** $140\frac{1}{4}$ **OR 140.25**

Solution: Since $xy = 4$ and $xy(x+y) = x^2y + xy^2 = 25$, we have $x + y = \frac{25}{4}$. Thus,

$$\begin{aligned} x^3y + x^2y^2 + xy^3 &= xy(x^2 + xy + y^2) \\ &= xy((x+y)^2 - xy) \\ &= 4 \cdot \left(\left(\frac{25}{4}\right)^2 - 4\right) \\ &= \boxed{\frac{561}{4}}. \end{aligned}$$

14. Shivani is planning a road trip in a car with special new tires made of solid rubber. Her tires are cylinders that have width 6 inches and have diameter 26 inches, but need to be replaced when the diameter is less than 22 inches. The tire manufacturer claims that 0.12π cubic inches of its tire will wear away with every single rotation. Assuming that the tire manufacturer is correct about the wear rate of their tires, and that the tire maintains its cylindrical shape and width (losing volume by reducing radius), how many revolutions can each tire make before she needs to replace it?

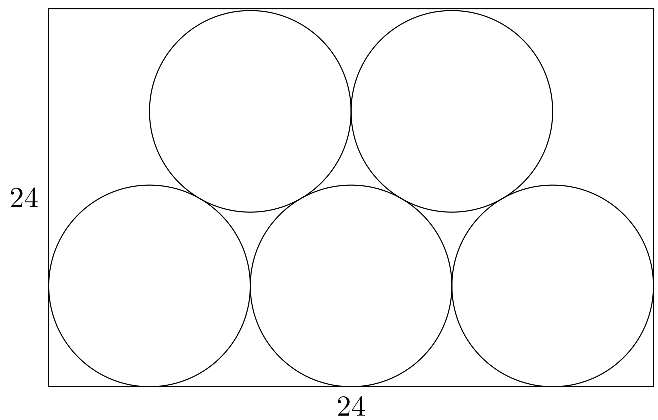
Answer: 2400

Solution: Notice that the volume of the tire that has worn away equals the number of times it rotated, multiplied by the rate of wear per rotation. Each tire is replaced before wearing off $((\frac{26}{2})^2\pi - (\frac{22}{2})^2\pi) \cdot 6 = 288\pi$ cubic inches, by which time it has made R revolutions. Then $288\pi = 0.12\pi \cdot R$, so $R = \boxed{2400}$.

15. What's the maximum number of circles of radius 4 that fit into a 24×15 rectangle without overlap?

Answer: 5

Solution:



A quick check shows that it is possible to fit 5 circles inside this rectangle; put three circles at the bottom of the rectangle, and there is space for two more circles in the space left over. Now, we will show that 6 circles is impossible. Note that the vertical distance between the centers of any two circles is at most $15-4-4=7$. The distance between any two centers is at least $4+4=8$, so the horizontal distance between the centers of any two circles is at least $\sqrt{8^2-7^2}=\sqrt{15}$. The leftmost and rightmost centers can be at most $24-4-4=16$ apart; as such, the total horizontal distance that the circles take if there are 6 circles is at least $5\cdot\sqrt{15}+8>24$. Thus, the answer is $\boxed{5}$.

16. Let $\{a_i\}$ for $1 \le i \le 10$ be a finite sequence of 10 integers such that for all odd i, $a_i = 1$ or -1, and for all even i, $a_i = 1, -1$, or 0. How many sequences $\{a_i\}$ exist such that $a_1+a_2+a_3+\cdots+a_{10}=0$?

Answer: 1052

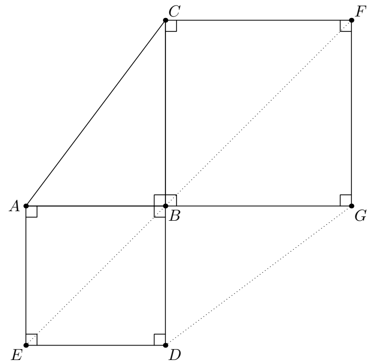
Solution: If k elements of the sequence equal 0, then k must be even in order for the sum to equal 0, so $k=0,2,4$. In that case, there must be $\frac{10-k}{2}$ each of 1's and -1's among the sequence. There are $\binom{5}{k}$ ways to pick which k elements of the sequence are 0 and $\binom{10-k}{\frac{10-k}{2}}$ ways to pick which elements of the sequence are 1's. Then the rest are -1's. Thus, the answer is the sum over $k=0,2,4$ over the binomial coefficient

$$\binom{5}{k}\binom{10-k}{\frac{10-k}{2}}.$$

We have

$$\binom{5}{0}\binom{10}{5}+\binom{5}{2}\binom{8}{4}+\binom{5}{4}\binom{6}{3}=252+700+100=\boxed{1052}.$$

17. Let $\triangle ABC$ be a right triangle with $m\angle B = 90^\circ$ such that AB and BC have integer side lengths. Squares $ABDE$ and $BCFG$ lie outside $\triangle ABC$. If the area of $\triangle ABC$ is 12, and the area of quadrilateral $DEFG$ is 38, compute the perimeter of $\triangle ABC$.



Answer: $10 + 2\sqrt{13}$

Solution: Let $AB = a$, $BC = b$. Then $ab = 2 \cdot A_{\triangle ABC} = 24$. Note that $38 = A_{DEFG} = \frac{a^2}{2} + \frac{b^2}{2} + \frac{ab}{2}$, as the interior of $DEFG$ is the union of half of the two squares and a right triangle with leg lengths a and b. Thus $a^2 + b^2 = 52$, and $\sqrt{a^2 + b^2} = \sqrt{52} = 2\sqrt{13}$ is the hypotenuse length of $\triangle ABC$. It follows that $(a, b) = (4, 6)$ or $(a, b) = (6, 4)$, and $P_{\triangle ABC} = \boxed{10 + 2\sqrt{13}}$.

18. What is the smallest positive integer x such that there exists an integer y with $\sqrt{x}+\sqrt{y} = \sqrt{1025}$?

Answer: 41

Solution: Note that $\sqrt{1025} = 5\sqrt{41}$. Subtracting $\sqrt{x}$ from both sides and squaring both sides gives $y = 1025 + x - 2 \cdot 5\sqrt{41x}$. Therefore, $\sqrt{41x}$ must be an integer, and $41 \mid x$. Then the smallest positive value of x is $\boxed{41}$, in which case $y = 16 \cdot 41$.

19. Let

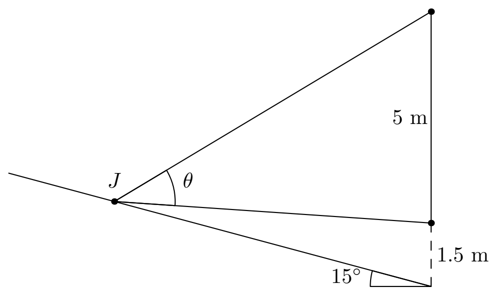
$$a = \underbrace{19191919\ldots1919}_{\text{19 is repeated 3838 times}}.$$

What is the remainder when a is divided by 13?

Answer: 6

Solution: Note that $13 \mid 191919$. Thus, $13 \mid \underbrace{19191919\ldots191900}_{\text{19 is repeated 3837 times}}$, since $3 \mid 3837$. However, $a = \underbrace{19191919\ldots191900}_{\text{19 is repeated 3837 times}} + 19$, so the remainder when a is divided by 13 is the same as the remainder when 19 is divided by 13, or $\boxed{6}$.

20. James is watching a movie at the cinema. The screen is on a wall and is 5 meters tall with the bottom edge of the screen 1.5 meters above the floor. The floor is sloped downwards at 15 degrees towards the screen. James wants to find a seat which maximizes his vertical viewing angle (depicted below as θ in a two dimensional cross section), which is the angle subtended by the top and bottom edges of the screen. How far back from the screen in meters (measured along the floor) should he sit in order to maximize his vertical viewing angle?



Answer: $\frac{\sqrt{39}}{2}$ **OR** $\sqrt{\frac{39}{4}}$

Solution: Imagine a 2D cross-section of the cinema containing James and perpendicular to the screen. Draw a circle through James, the top edge of the screen and the bottom edge of the screen. The vertical viewing angle is maximized when the circle is tangent to the floor. This is because the radius of the circle is minimized when that happens and given that the viewing angle is twice the angle of the arc that starts from the top of the screen to the bottom of the screen, since the distance from the top edge of the screen to the bottom edge of the screen is fixed, the angle measurement of the arc is maximized when the radius of the circle is minimized.

Then by the power of the point theorem the distance is $\sqrt{1.5 \times 6.5} = \boxed{\frac{\sqrt{39}}{2}}$.

What follows is a proof that the radius of the circle is minimized when the circle is tangent to the floor. Let ℓ be the perpendicular bisector of the top of the screen and the bottom of the screen, let O be a point on this line, let C be the circle with center O that passes through the top and bottom edges of the screen and let A be the arc starting from the point closest to and passes through the bottom of the screen and the top of the screen, in that order. Note that the vertical viewing angle is half the measure of the arc A. Now imagine moving the point O toward James starting from behind the screen. As we do this, the arc A gets larger, the radius of C gets larger and the angle subtended by the screen from a point on A gets smaller. Then the maximum angle occurs when the arc A first touches the floor, which is when the circle C is tangent to the floor.

Note: the circle through the top and bottom of the screen that is radius 2.5 meters does not intersect the floor since the floor is 1.5 meters above the ground. If it were less than ~ 0.8 meters, then it would and the answer would not be obtained via the power of the point.